- Money has a time value because a unit of money received today is worth more than a unit of money to be received tomorrow.

2. INTEREST RATES: INTERPRETATION

Interest rates can be interpreted in three ways.

1) **Required rates of return:** It refers to the minimum rate of return that an investor must earn on his/her investment.

2) **Discount rates:** Interest rate can be interpreted as the rate at which the future value is discounted to estimate its value today.

3) **Opportunity cost:** Interest rate can be interpreted as the opportunity cost which represents the return forgone by an investor by spending money today rather than saving it. For example, an investor can earn 5% by investing $1000 today. If he/she decides to spend it today instead of investing it, he/she will forgo earning 5%.

Interest rate = r = Real risk-free interest rate + Inflation premium + Default risk premium + Liquidity premium + Maturity premium

- **Real risk-free interest rate:** It reflects the single-period interest rate for a completely risk-free security when no inflation is expected.
- **Inflation premium:** It reflects the compensation for expected inflation.
  Nominal risk-free rate = *Real risk-free interest rate + Inflation premium*
  - E.g. interest rate on a 90-day U.S. Treasury bill (T-bill) refers to the nominal interest rate.
- **Default risk premium:** It reflects the compensation for default risk of the issuer.
- **Liquidity premium**: It reflects the compensation for the risk of loss associated with selling a security at a value less than its fair value due to high transaction costs.
- **Maturity premium**: It reflects the compensation for the high interest rate risk associated with long-term maturity.

3. THE FUTURE VALUE OF A SINGLE CASH FLOW

The future value of cash flows can be computed using the following formula:

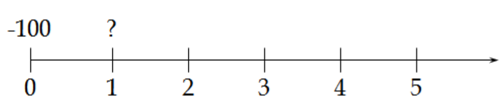
$$FV_N = PV(1+r)^N$$

where,

PV = Present value of the investment
FV_N = Future value of the investment N periods from today
Pmt = Per period payment amount
N = Total number of cash flows or the number of a specific period
r = Interest rate per period
$(1 + r)^N$ = FV factor

Example:

Suppose,
PV = \$100, N = 1, r = 10%. Find FV.

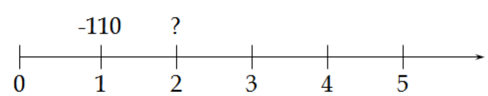


$$FV_1 = 100(1+0.10)^1 = 110$$

- The interest rate earned each period on the original investment (i.e. principal) is called simple interest e.g. \$10 in this example.

Simple interest = Interest rate × Principal

If at the end of year 1, the investor decides to extend the investment for a second year. Then the amount accumulated at the end of year 2 will be:



$$FV_2 = 100(1+0.10)(1+0.10) = 121$$

or

$$FV_2 = 100(1+0.10)^2 = 121$$

- Note that $FV_2 > FV_1$ because the investor earns ***interest on the interest*** that was earned in previous years (i.e. due to compounding of interest) in addition to the interest earned on the original principal amount.
- The effect of compounding increases with the increase in interest rate i.e. for a given compounding period (e.g. annually), the FV for an investment with 10% interest rate will be > FV of investment with 5% interest rate.

NOTE:

- For a given interest rate, the more frequently the compounding occurs (i.e. the greater the N), the greater will be the future value.
- For a given number of compounding periods, the higher the interest rate, the greater will be the future value.

Important to note:

Both the interest rate (r) and number of compounding periods (N) must be compatible i.e. if N is stated in months then r should be 1-month interest rate, un-annualized.

Practice: Example 1, 2 & 3, Volume 1, Reading 6.

3.1 The Frequency of Compounding

With more than one compounding period per year,

$$FV_N = PV\left(1+\frac{r_s}{m}\right)^{m\times N}$$

where,

r_s = stated annual interest rate
m = number of compounding periods per year
N = Number of years

Stated annual interest rate: It is the quoted interest rate that does not take into account the compounding within a year.

Stated annual interest rate **=** *Periodic interest rate × Number of compounding periods per year*

Periodic interest rate **= r_s / m =** *Stated annual interest rate / Number of compounding periods per year*

where,

Number of compounding periods per year = Number of compounding periods in one year × number of years = m×N

NOTE:

The more frequent the compounding, the greater will be the future value.

Example:

Suppose,

A bank offers interest rate of 8% compounded quarterly on a CD with 2-years maturity. An investor decides to invest $100,000.

- PV = $100,000
- N = 2
- r_s = 8% compounded quarterly
- m = 4
- r_s / m = 8% / 4 = 2%
- mN = 4 (2) = 8

FV = $100,000 (1.02)^8 = $117,165.94

Practice: Example 4, 5 & 6, Volume 1, Reading 6.

3.2 Continuous Compounding

When the number of compounding periods per year becomes infinite, interest rate is compounded continuously. In this case, FV is estimated as follows:

$$FV_N = PVe^{r_s \times N}$$

where,

e = 2.7182818

- The continuous compounding generates the ***maximum*** future value amount.

Example:

Suppose, an investor invests $10,000 at 8% compounded continuously for two years.

FV = $10,000 $e^{0.08\,(2)}$ = $11,735.11

3.3 Stated and Effective Rates

Periodic interest rate = Stated annual interest rate / Number of compounding periods in one year (i.e. m)

E.g. m = 4 for quarterly, m = 2 for semi-annually compounding, and m = 12 for monthly compounding.

Effective (or equivalent) annual rate (EAR = EFF %): It is the annual rate of interest that an investor ***actually*** earns on his/her investment. It is used to compare investments with different compounding intervals.

EAR (%) = (1 + Periodic interest rate)^m – 1

- Given the EAR, periodic interest rate can be calculated by reversing this formula.
  Periodic interest rate = (EAR(%) + 1)$^{1/m}$ –1

For example, EAR% for 10% semiannual investment will be:

m = 2
stated annual interest rate = 10%
EAR = (1 + (0.10 / 2)) ^{2} – 1 = 10.25%

- This implies that an investor should be indifferent between receiving 10.25% ***annual*** interest rate and receiving 10% interest rate compounded ***semiannually***.

EAR with continuous compounding:

$$\mathbf{EAR = e^{r_s} - 1}$$

- Given the EAR, periodic interest rate can be calculated as follows:
  $$EAR + 1 = e^{r_s}$$
- Now taking the natural logarithm of both sides we have:
  $\ln(EAR + 1) = \ln e^{r_s}$ ➔ (since ln e = 1)
  $\ln(EAR + 1) = r_s$

Now taking the natural logarithm of both sides we have:

$$EAR + 1 = \ln e^{r_s} \rightarrow \text{(since ln e = 1)}$$
$$EAR + 1 = r_s$$

NOTE:

Annual percentage rate (APR): It is used to measure the cost of borrowing stated as a yearly rate.

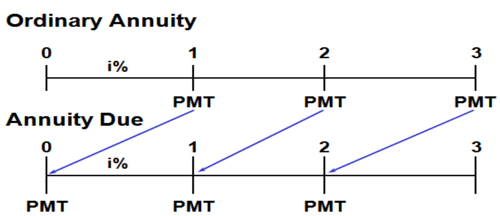
APR = Periodic interest rate × Number of payments periods per year

4. THE FUTURE VALUE OF A SERIES OF CASH FLOWS

Annuity:

Annuities are ***equal*** and finite set of periodic outflows/ inflows at ***regular intervals*** e.g. rent, lease, mortgage, car loan, and retirement annuity payments.

- **Ordinary Annuity:** Annuities whose payments begin at the ***end*** of each period i.e. the 1st cash flow occurs one period from now (t = 1) are referred to as ordinary annuity e.g. mortgage and loan payments.
- **Annuity Due:** Annuities whose payments begin at the ***start*** of each period i.e. the 1st cash flow occurs immediately (t = 0) are referred to as annuity due e.g. rent, insurance payments.



Present value and future value of Ordinary Annuity:

The <u>**future value**</u> of an ***ordinary annuity*** stream is calculated as follows:

$$\mathbf{FV_{OA}} = Pmt\left((1+r)^{N-1} + (1+r)^{N-2} + \ldots + (1+r)^1 + (1+r)^0\right)$$

$$FV_{OA} = \sum_{t=1}^{N} Pmt_t(1+r)^{N-t} = Pmt\left[\frac{(1+r)^N - 1}{r}\right]$$

$$\text{FV annuity factor} = \left[\frac{(1+r)^N - 1}{r}\right]$$

where,

Pmt = Equal periodic cash flows
r = Rate of interest
N = Number of payments, one at the end of each period (ordinary annuity).

The <u>**present value**</u> of an ***ordinary annuity*** stream is calculated as follows:

$$PV_{OA} = \sum_{t=1}^{N} \frac{Pmt}{(1+r)^t}$$
$$= Pmt_1/(1+r)^{N-1} + Pmt_2/(1+r)^{N-2} + \cdots + Pmt_N/(1+r)^N)$$

Or

$$PV_{OA} = \sum_{t=1}^{N} \frac{Pmt}{(1+r)^t} = Pmt\left[\frac{1 - \frac{1}{(1+r)^N}}{r}\right]$$

Present value and future value of Annuity Due:

The present value of an ***annuity due*** stream is calculated as follows (section 6).

$$PV_{AD} = Pmt\left[\frac{1 - \frac{1}{(1+r)^{(N-1)}}}{r}\right] + Pmt\ at\ t = 0$$

Or

$$PV_{AD} = Pmt\left[\frac{1 - \frac{1}{(1+r)^{N}}}{r}\right](1+r)$$

$$PV_{AD} = PV_{OA} + Pmt$$

where,

Pmt = Equal periodic cash flows
r = Rate of interest
N = Number of payments, one at the beginning of each period (annuity due).

- It is important to note that PV of annuity due > PV of ordinary annuity.

NOTE:

PV of annuity due can be calculated by setting calculator to "BEGIN" mode and then solve for the PV of the annuity.

The future value of an ***annuity due*** stream is calculated as follows:

$$FV_{AD} = Pmt\left[\frac{(1+r)^N - 1}{r}\right](1+r)$$

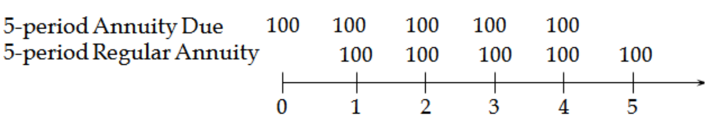
Or

$FV_{AD} = FV_{OA} \times (1 + r)$

- It is important to note that FV of annuity due > FV of ordinary annuity.

Example:

Suppose a 5-year, $100 annuity with a discount rate of 10% annually.

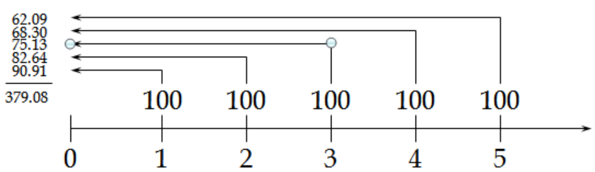


Calculating Present Value for Ordinary Annuity:

$$PV_{OA} = \frac{100}{(1.10)^1} + \frac{100}{(1.10)^2} + \frac{100}{(1.10)^3} + \frac{100}{(1.10)^4} + \frac{100}{(1.10)^5}$$
$$= 379.08$$

Or

$$PV_{OA} = Pmt\left[\frac{1 - \frac{1}{(1.10)^5}}{0.10}\right] = 379.08$$

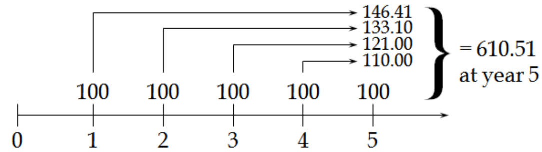


Using a Financial Calculator: N= 5; PMT = –100; I/Y = 10; FV=0; CPT➔PV = $379.08

Calculating Future Value for Ordinary Annuity:
FV_{OA}
$=100(1.10)^4+100(1.10)^3+100(1.10)^2+100(1.10)^1+100=610.51$

Or

$$FV_{OA} = 100\left[\frac{(1.10)^5 - 1}{0.10}\right] = 610.51$$



Using a Financial Calculator: N= 5; PMT = -100; I/Y = 10; PV=0; CPT➔FV = $610.51

Annuity Due: An annuity due can be viewed as = $100 lump sum today + Ordinary annuity of $100 per period for four years.

Calculating Present Value for Annuity Due:

$$PV_{AD} = 100\left[\frac{1 - \frac{1}{(1.10)^{(5-1)}}}{0.10}\right] + 100 = 416.98$$

Calculating Future value for Annuity Due:

$$FV_{AD} = 100\left[\frac{(1.10)^5 - 1}{0.10}\right](1.10) = 671.56$$

Practice: Example 7, 11, 12 & 13, Volume 1, Reading 6.

4.2 Unequal Cash Flows

Time	Cash Flow ($)	Future Value at Year 5	
$t = 1$	1,000	$\$1{,}000(1.05)^4 =$	$1,215.51
$t = 2$	2,000	$\$2{,}000(1.05)^3 =$	$2,315.25
$t = 3$	4,000	$\$4{,}000(1.05)^2 =$	$4,410.00
$t = 4$	5,000	$\$5{,}000(1.05)^1 =$	$5,250.00
$t = 5$	6,000	$\$6{,}000(1.05)^0 =$	$6,000.00
		Sum	= $19,190.76

Source: Table 2.

FV at t = 5 can be calculated by computing FV of each payment at t = 5 and then adding all the individual FVs e.g. as shown in the table above:

FV of cash flow at t =1 is estimated as
FV = $1,000 (1.05) ^{4} = $1,215.51

5.1 Finding the Present Value of a Single Cash Flow

The present value of cash flows can be computed using the following formula:

$$PV = \frac{FV_N}{(1+r)^N}$$

- **The PV factor =** 1 / (1 + r) ^{N}; It is the reciprocal of the FV factor.

NOTE:

- For a given discount rate, the greater the number of periods (i.e. the greater the N), the smaller will be the present value.
- For a given number of periods, the higher the discount rate, the smaller will be the present value.

Practice: Example 8 & 9, Volume 1, Reading 6.

5.1 The Frequency of Compounding

With more than one compounding period per year,

$$PV = FV_N \left(1+\frac{r_s}{m}\right)^{-mN}$$

where,

r_s = stated annual interest rate
m = number of compounding periods per year
N = Number of years

Practice: Example 10, Volume 1, Reading 6.

6.2 The Present Value of an Infinite Series of Equal Cash Flows i.e. Perpetuity

Perpetuity: It is a set of infinite periodic outflows/ inflows at regular intervals and the 1st cash flow occurs one period from now (t=1). It represents a perpetual annuity e.g. preferred stocks and certain government bonds make equal (level) payments for an indefinite period of time.

PV = Pmt / r

This formula is valid only for perpetuity with level payments.

Example:

Suppose, a stock pays constant dividend of $10 per year, the required rate of return is 20%. Then the PV is calculated as follows.

PV = $10 / 0.20 = $50

Practice: Example 14, Volume 1, Reading 6.

6.3 Present Values Indexed at Times Other than t =0

Suppose instead of t = 0, first cash flow of $6 begin at the end of year 4 (t = 4) and continues each year thereafter till year 10. The discount rate is 5%.

- It represents a seven-year Ordinary Annuity.

a) First of all, we would find PV of an annuity at t = 3 i.e.

N = 7, I/Y = 5, Pmt = 6, FV = 0, CPT➔PV_3 = $34.72

b) Then, the PV at t = 3 is again discounted to t = 0.

N = 3, I/Y = 5, Pmt = 0, FV = 34.72, CPT➔ PV_0 = $29.99

Practice: Example 15, Volume 1, Reading 6.

- An annuity can be viewed as the difference between two perpetuities with equal, level payments but with different starting dates.

Example:

- Perpetuity 1: $100 per year starting in Year 1 (i.e. 1st payment is at t =1)
- Perpetuity 2: $100 per year starting in Year 5 (i.e. 1st payment is at t = 5)
- A 4-year Ordinary Annuity with $100 payments per year and discount rate of 5%.

4-year Ordinary annuity = Perpetuity 1 – Perpetuity 2

PV of 4-year Ordinary annuity = PV of Perpetuity 1 – PV of Perpetuity 2

i. PV_0 of Perpetuity 1 = $100 / 0.05 = $2000
ii. PV_4 of Perpetuity 2 = $100 / 0.05 = $2000
iii. PV_0 of Perpetuity 2 = $2000 / $(1.05)^4$ = $1,645.40
iv. PV_0 of Ordinary Annuity = PV_0 of Perpetuity 1 - PV_0 of Perpetuity 2
= $2000 - $1,645.40
= $354.60

6.4 The Present Value of a Series of Unequal Cash Flows

Suppose, cash flows for Year 1 = $1000, Year 2 = $2000, Year 3 = $4000, Year 4 = $5000, Year 5 = 6,000.

A. ***Using the calculator's "CFLO" register***, enter the cash flows

- CF_0 = 0
- CF_1 = 1000
- CF_2 = 2000
- CF_3 = 4000

- CF_4 = 5000
- CF_5 = 6000

Enter I/YR = 5, ➔ press NPV➔NPV or PV = $15,036.46

Or

B. PV can be calculated by computing PV of each payment separately and then adding all the individual PVs e.g. as shown in the table below:

Time Period	Cash Flow ($)	Present Value at Year 0	
1	1,000	$\$1{,}000(1.05)^{-1}$ =	$952.38
2	2,000	$\$2{,}000(1.05)^{-2}$ =	$1,814.06
3	4,000	$\$4{,}000(1.05)^{-3}$ =	$3,455.35
4	5,000	$\$5{,}000(1.05)^{-4}$ =	$4,113.51
5	6,000	$\$6{,}000(1.05)^{-5}$ =	$4,701.16
		Sum	= $15,036.46

Source: Table 3.

7.1 Solving for Interest Rates and Growth Rates

An interest rate can be viewed as a growth rate (g).

$$g = (FV_N/PV)^{1/N} - 1$$

Practice: Example 17 & 18, Volume 1, Reading 6.

7.2 Solving for the Number of Periods

$$N = (\ln(FV / PV)) / \ln(1 + r)$$

Suppose, FV = $20 million, PV = $10 million, r = 7%. Number of years it will take $10 million to double to $20 million is calculated as follows:

N = ln (20 million / 10 million) / ln (1.07) = 10.24 ≈ 10 years

7.3 Solving for the Size of Annuity Payments

Annuity Payment = Pmt = $\frac{\text{PV}}{\text{PV Annuity Factor}}$

Suppose, an investor plans to purchase a $120,000 house; he made a down payment of $20,000 and borrows the remaining amount with a 30-year fixed-rate mortgage with monthly payments.

- The amount borrowed = $100,000
- 1st payment is due at t = 1
- Mortgage interest rate = 8% compounding monthly.
  - o PV = $100,000
  - o r_s = 8%
  - o m = 12
  - o Period interest rate = 8% / 12 = 0.67%
  - o N = 30
  - o mN = 12 × 30 = 360

$$PV\ annuity\ factor = \frac{1 - \frac{1}{\left[1+\left(\frac{r_s}{m}\right)\right]^{mN}}}{\frac{r_s}{m}}$$

$$= \frac{1 - \frac{1}{(1.006667)^{360}}}{0.006667} = 136.283494$$

Pmt = PV / Present value annuity factor
= $100,000 / 136.283494 = $733.76

- Thus, the $100,000 amount borrowed is equivalent to 360 monthly payments of $733.76.

IMPORTANT Example:

Calculating the projected annuity amount required to fund a future-annuity inflow.

Suppose Mr. A is 22 years old. He plans to retire at age 63 (i.e. at t = 41) and at that time he would like to have a retirement income of $100,000 per year for the next 20 years. In addition, he would save $2,000 per year for the next 15 years (i.e. t = 1 to t = 15) by investing in a bond mutual fund that will generate 8% return per year on average.

So, to meet his retirement goal, the total amount he needs to save each year from t = 16 to t = 40 is estimated as follows:

Calculations:

It should be noted that:

PV of savings (outflows) **must equal** PV of retirement income (inflows)

a) At t =15, Mr. A savings will grow to:

$$FV = 2000\left[\frac{(1.08)^{15} - 1}{0.08}\right] = \$54{,}304.23$$

b) The total amount needed to fund retirement goal i.e. PV of retirement income at t = 15 is estimated using two steps:

i. We would first estimate PV of the annuity of $100,000 per year for the next 20 years at t = 40.

$$PV_{40} = \$100{,}000\left[\frac{1 - \frac{1}{(1.08)^{20}}}{0.08}\right] = \$981{,}814.74$$

ii. Now discount PV $_{40}$ back to t = 15. From t = 40 to t = 15 ➔ total number of periods (N) = 25.

N = 25, I/Y = 8, Pmt = 0, FV = $981,814.74, CPT➔ PV = $143,362.53

- Since, PV of savings (outflows) must equal PV of retirement income (inflows)
The total amount he needs to save each year (from t = 16 to t = 40) i.e.

Annuity = Amount needed to fund retirement goals - Amount already saved
= $143,362.53 - $54,304.23 = $89,058.30

- The annuity payment per year from t = 16 to t = 40 is estimated as:
  Pmt = PV / Present value annuity factor
  - PV of annuity = $89,058.30
  - N = 25
  - r = 8%

$$PV\ annuity\ factor = \left[\frac{1-\frac{1}{(1.08)^{25}}}{0.08}\right] = 10.674776$$

Annuity payment = pmt = $89,058.30 / 10.674776
= $8,342.87

Source: Example 21, Volume 1, Reading 6.

7.4 Equivalence Principle

Principle 1: A lump sum is equivalent to an annuity i.e. if a lump sum amount is put into an account that generates a stated interest rate for all periods, it will be equivalent to an annuity.

Examples include amortized loans i.e. mortgages, car loans etc.

Example:

Suppose, an investor invests $4,329.48 in a bank today at 5% interest for 5 years.

$$Annuity\ payments = \frac{PV}{\frac{1-[(1/(1+r)^{N})]}{r}} = \frac{\$4{,}329.48}{\frac{1-[(1/(1+1.05)^{5})]}{0.05}} = \$1{,}000$$

- Thus, a lump sum initial investment of $4,329.48 can generate $1,000 withdrawals per year over the next 5 years.
- $1,000 payment per year for 5 years represents a 5-year ordinary annuity.

Principle 2: An annuity is equivalent to the FV of the lump sum.

For example from the example above stated.
FV of annuity at t = 5 is calculated as:

N = 5, I/Y = 5, Pmt = 1000, PV = 0,
CPT➔FV = $5,525.64
And the PV of annuity at t = 0 is:

N = 5, I/Y = 5, Pmt = 0, FV = 5,525.64,
CPT➔ PV =$4,329.48.

7.5 The Cash Flow Additivity Principle

The Cash Flow Additivity Principle: The amounts of money indexed at the same point in time are additive.

Example:

Interest rate = 2%.
Series A's cash flows:

t = 0 ➔ 0
t = 1 ➔ $100
t = 2 ➔ $100

Series B's cash flows:

t = 0 ➔ 0
t = 1 ➔ $200
t = 2 ➔ $200

- Series A's FV = $100 (1.02) + $100 = $202
- Series B's FV = $200 (1.02) + $200 = $404
- FV of (A + B) = $202 + $404 = $606

FV of (A + B) can be calculated by adding the cash flows of each series and then calculating the FV of the combined cash flow.

- At t = 1, combined cash flows = $100 + $200 = $300
- At t = 2, combined cash flows = $100 + $200 = $300
  Thus, FV of (A+ B) = $300 (1.02) + $300 = $606

Example:

Suppose,

Discount rate = 6%
At t = 1 → Cash flow = $4
At t = 2 → Cash flow = $24

It can be viewed as a $4 annuity for 2 years and a lump sum of $20.

N = 2, I/Y = 6, Pmt = 4, FV = 0,
CPT➔ PV of $4 annuity = $7.33

N = 2, I/Y = 6, Pmt = 0, FV = 20,
CPT➔PV of lump sum = $17.80

Total = $7.33 + $17.80 = $25.13

Practice: End of Chapter Practice Problems for Reading 6.